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Critically finite maps on projective spaces (II)

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1 Introduction

Critically finite maps provide interesting examples of complex dynamics on projective spaces which can be fairly well analyzed. For one dimensional case such maps are first investigated by Thurston. A holomorphic map f from the Riemann sphere $\mathbb{P}^1$ onto itself, i.e., a rational function of one variable, is said to be critically finite if every critical point of f is (pre-)periodic. It is called strictly critically finite if every critical point is preperiodic but not periodic. Thurston's theorem asserts that the Julia set for a strictly critically finite map coincides with the whole $\mathbb{P}^1$.

Generalizations of critically finite maps on projective spaces of general dimension were first studied by Fornæss-Sibony [FS1, FS2, FS3] (see also [U1]). In [U3] we showed, for the case of dimension 2, that strictly critically finite maps have empty Fatou set. Further, Jonsson [J] showed that the support of the invariant measure is all of $\mathbb{P}^2$.

In this article we propose a definition of strictly critically finite map and develop the method introduced in [U3, U4, U5] further. This may be regarded as a generalization of Thurston's theorem for strictly critically finite maps on projective spaces of any dimension.

2 Definitions and main results

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$ and let C denote its critical set. The map f is a d^n-fold branched covering over $\mathbb{P}^n$ whose branch locus lies over the set $f(C)$. For every $i \geq 1$, the critical set of the iterate f^i is $\bigcup_{k=0}^{i-1} f^{-k}(C)$ and f^i is a d^{ni}-fold branched covering whose branch locus lies over $\bigcup_{k=1}^{i} f^k(C)$.

We define the postcritical set of f by $D = \bigcup_{i\geq 1} f^i(C)$. We will say that f is critically finite if D is an algebraic subset of $\mathbb{P}^n$. For such a map f, the iterates $f^i : \mathbb{P}^n \to \mathbb{P}^n \quad (i \geq 1)$ are branched covering whose branch locus lies only over the postcritical set D.

We will say that f is strictly critically finite if the branching order of the map f^i is everywhere bounded by some number independent of i. We note that, this definition of strictly critically finite map reduces to the original in the case of dimension one, and coincides with the definition of n-critically maps given by Jonsson [J] (see also [Rn]).

Our main result is the following theorem.

Theorem 2.1 *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a strictly critically finite map and let K be a compact connected subset of $\mathbb{P}^n$ containing at least two points. Then there is no subsequence of $\{f^i\}$ that is uniformly convergent on K.*

As consequences of this theorem, we have the following theorems.

Theorem 2.2 *For a strictly critically finite map $f : \mathbb{P}^n \to \mathbb{P}^n$, all periodic points of are repelling. Further the set of all (repelling) periodic points is dense in $\mathbb{P}^n$.*

Theorem 2.3 *If $f : \mathbb{P}^n \to \mathbb{P}^n$ is a strictly critically finite map, then, for any point $a \in \mathbb{P}^n$, the set $\bigcup_{j=1}^{\infty} f^{-j}(a)$ is dense in $\mathbb{P}^n$. There exists no closed subset of $\mathbb{P}^n$ that is backward invariant under f except for the empty set and the whole $\mathbb{P}^n$.*

In the proof of the theorem, we will use the concepts of Fatou maps and branched coverings.

3 Outline of the proof

3.1 Fatou maps

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a holomorphic map of degree $d \geq 2$ and let $\varphi : X \to \mathbb{P}^n$ be a holomorphic map from a connected complex analytic space X into $\mathbb{P}^n$. We say that φ is a Fatou map for f if the sequence $\{f^i \circ \varphi\}_i$ is a normal family. This may be considered as a generalization of the Fatou set, and admits a characterization using the Green function, similar to that of Fatou sets.

For a holomorphic map $\varphi : X \to \mathbb{P}^n$, we say that a holomorphic map $\psi : X \to \mathbb{P}^n$ is a holomorphic lift of φ by an iterate f^i of f if $f^i \circ \psi = \varphi$ holds. We note that, when X is an open subset of $\mathbb{P}^n$ and φ is the inclusion map, such a lift ψ is a branch on X of the inverse of f^i.

If $\varphi : X \to \mathbb{P}^n$ is a holomorphic map, then the set of all possible lifts $\psi : X \to \mathbb{P}^n$ by some iterate f^i forms a normal family (Theorem 2.1 in

[U3]). Further, we can prove easily that the limit of any locally uniformly convergent sequence of lifts is a Fatou map.

If X is compact, then there exists no nonconstant Fatou map from X. The following theorem generalizes this fact.

Theorem 3.1 *Let φ be a holomorphic map from an irreducible complex space X of positive dimension into $\mathbb{P}^n$. Suppose that, for every point $p_0 \in \mathbb{P}^n$, there exists a neighborhood U of p_0 such that either $\varphi^{-1}(U)$ is empty or every connected component of $\varphi^{-1}(U)$ is relatively compact in X. Then φ is not a Fatou map for any (not necessarily critically finite) holomorphic map $f : \mathbb{P}^n \to \mathbb{P}^n$ of degree > 2.*

It turns out that a strictly critically finite map admits no nonconstant Fatou map.

To prove the main result, we suppose the existence of a non-trivial connected compact set K and construct a Fatou map that contradicts the above theorem.

3.2 Branched coverings

Let B be a connected and locally connected Hausdorff space. A continuous map η from a Hausdorff space Y onto B is called an unbranched covering if for any point $b \in B$ there is a connected neighborhood V of b such that each connected component of $\eta^{-1}(V)$ is mapped homeomorphically onto V.

Now let A be a connected complex space. A holomorphic map ξ from a connected complex space X onto A is called a branched covering over A, if the following condition is satisfied: For any point $a \in A$, there exists a neighborhood U of a such that the restriction of ξ to each connected component of $\xi^{-1}(U)$ is a finite proper map. Let D be an analytic subset of A. A branched covering $\xi : X \to A$ will be called a D-branched covering if the restriction of ξ to $X \setminus \xi^{-1}(D)$ is an unbranched covering over $A \setminus D$.

If f is a strictly critically finite map, then the iterates $f^i : \mathbb{P}^n \to \mathbb{P}^n$ ($i = 1, 2, \ldots$) constitute a family of coverings that are branched only over the postcritical set D. The following theorem asserts that we can construct a branched covering that dominates this family of coverings.

Theorem 3.2 *Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a strictly critically finite map with postcritical set D. Then there exists a D-branched covering $\xi : X \to \mathbb{P}^n$ with the following property: For any $i \geq 1$ and any pair of points $p \in \mathbb{P}^n$ and $x \in X$ with $f^i(p) = \xi(x)$, there exists a branched covering map $\varphi : X \to \mathbb{P}^n$ such that $f^i \circ \varphi = \xi$ and that $\varphi(x) = p$.*

This is a consequence of the following lemma that deals with a general situation of families of branched coverings.

Lemma 3.3 *Let $\xi_\lambda : X_\lambda \to A$ ($\lambda \in \Lambda$) be a family of D-branched coverings. Suppose that there is a constant m such that, for any $\lambda \in \Lambda$ and for any point $x \in X_\lambda$, the branching order* $\mathrm{ord}(\xi_\lambda, x)$ *is bounded by m.*

Then there exists a normal D-branched covering $\widehat{\xi} : \widehat{X} \to A$ with the following property: For any $\lambda \in \Lambda$ and any pair of points $x \in X_\lambda$ and $\widehat{x} \in \widehat{X}$ with $\xi_\lambda(x) = \xi(\widehat{x})$, there is a $\xi_\lambda^{-1}(D)$-branched covering $\psi : \widehat{X} \to X_\lambda$ such that $\xi_\lambda \circ \psi = \widehat{\xi}$ and that $\psi(\widehat{x}) = x$. Further there exists a minimal such D-branched covering determined uniquely up to isomorphism.

3.3 Proof of the main theorem

Let $f : \mathbb{P}^n \to \mathbb{P}^n$ be a strictly critically finite map. Suppose that there exists a connected compact subset K of $\mathbb{P}^n$ containing at least two points and a subsequence of the iterates f^i uniformly convergent on K, and let $h : K \to \mathbb{P}^n$ be the limit of the sequence. We take the branched covering $\xi : X \to \mathbb{P}^n$ that dominates the iterates f^i (Theorem 3.2).

First we show that the map h is nonconstant. We let $\hat{K}$ be a connected component of $\xi^{-1}(h(K))$. We choose a sequence $\{\psi_\nu\}_\nu$ of lifts $\psi_\nu : X \to \mathbb{P}^n$ of ξ by some $f^{i(\nu)}$ that converges to a holomorphic map $\psi_* : X \to \mathbb{P}^n$. The sequence $\{\psi_\nu\}_\nu$ can be so chosen that ψ_* and ξ coincide on $\hat{K}$. Let Z be the connected component of the analytic set $\{x \in X \mid \psi_*(x) = \xi(x)\}$ containing $\hat{K}$. Then the map $\psi_*|Z = \xi|Z : Z \to \mathbb{P}^n$ satisfies the condition of Theorem 3.1 and this contradicts that this map is a Fatou map.

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